Week 2 Worksheet
137A Review; QM in 3D (and some spin)

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Exercise 1. A Hilbert space is mathematically defined as a *complete* vector space with an inner product. A vector space with an inner product is **complete** if it includes not only all finite sums of vectors in a basis, but also all limits of convergent sequences, i.e. given a sequence (v_n) of vectors in the Hilbert space, v is the **limit** of the sequence if $\lim_{n\to\infty} \|v_n - v\| = 0$, where $\|v\| = \sqrt{v \cdot v}$.

a) Consider a Hilbert space $\mathcal{H}$ that consists of all functions $\psi(x)$ such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, \mathrm{d}x < \infty.$$

Show that there are functions in $\mathcal{H}$ for which $\hat{x}\psi(x) = x\psi(x)$ is not in $\mathcal{H}$.

b) Consider the function space Ω in $\mathcal{H}$ which consists of all $\varphi(x)$ that satisfy the set of conditions

$$\int_{\infty}^{\infty} |\varphi(x)|^2 (1 + |x|)^n \, \mathrm{d}x < \infty,$$

for any $n \in \{0, 1, 2, \ldots\}$. Show that for any $\varphi(x)$ in Ω, $\hat{x}\varphi(x)$ is also in Ω. Ω is called the **nuclear** space.

c) The **extended** space $\Omega^\times$ consists of those functions $\chi(x)$ which satisfy

$$(\chi, \varphi) = \int_{-\infty}^{\infty} \chi^*(x)\varphi(x) \, \mathrm{d}x < \infty,$$

for any φ in Ω, where (,) is the inner product on $\mathcal{H}$. Which of the following functions belong to Ω, to $\mathcal{H}$, and/or to $\Omega^\times$?

Remark. The collection $(\Omega, \mathcal{H}, \Omega^\times)$ is called "rigged Hilbert space," and this is a rigorous way to include all the formalism (e.g. eigenvectors of position are delta functions, and hence can't belong to an L^2 space) into the Hilbert space formulation of quantum mechanics. Note that $\Omega \subset \mathcal{H} \subset \Omega^\times$ (it's easy to see this once you realize $\mathcal{H} = \mathcal{H}^\times$). Also, note that in order to sit in Ω, functions must vanish faster than any power of x as $|x| \to \infty$. Thus, as long as functions don't diverge at ∞ more strongly than any power of $|x|$, they are in $\Omega^\times$. For more details, see Ballentine *Quantum Mechanics*, Chapter 1.

i) $\sin(x)$
ii) $\sin(x)/x$
iii) $x^2 \cos(x)$
iv) e^{-ax}, $a > 0$.
v) $\dfrac{\ln(1+|x|)}{1+|x|}$
vi) e^{-x^2}
vii) $x^4 e^{-|x|}$

Exercise 2. We say an operator on a Hilbert space is **hermitian** if its action on vectors is the same as that of its hermitian adjoint, i.e. for all v in the Hilbert space, $Qv = Q^\dagger v$. We say it is **self-adjoint** if in addition the domains of the two operators Q and $Q^\dagger$ are the same. Note that in a finite-dimensional Hilbert space, these two notions are identical (why?). It turns out that in order to represent observables, operators must be self-adjoint, not just hermitian.

a) What is the Hilbert space (of functions of position) associated to the solutions of the time-independent, 1-dimensional Schrödinger equation with hamiltonian $H = p^2/2m$ (you need to give a vector space *and* an inner product)?

b) Are the eigenvectors of position in this Hilbert space? What about momentum eigenvectors?

c) If we suppose that the domain of p is the entire Hilbert space from (a), is p hermitian here? What about self-adjoint?

d) Consider now the infinite square well on $[0, a]$. Given the form of the momentum operator $p = -i\hbar\, \mathrm{d}/\mathrm{d}x$, we could define its domain as the set of all L^2 functions $f(x)$ on $[0, a]$ such that $f(0) = 0$ and $f(a) = 0$. Show that in this case p is hermitian.[1] Is it also self-adjoint?

e) Suppose now we extend the domain of p to include functions that satisfy $f(a) = \lambda f(0)$, for some $\lambda \in \mathbb{C}$. What condition must we then impose on the domain of $p^\dagger$ such that $p^\dagger$ be hermitian? For what values of λ is p self-adjoint?

f) Instead of the infinite square well, consider a semi-infinite interval $0 \le x < \infty$. Is there a self-adjoint momentum operator in this case?

Exercise 3. Solve the eigenvalue problem for the 3-D isotropic, harmonic oscillator, whose hamiltonian is $H = p^2/2m + m\omega^2 x^2/2$, where $p^2 = \mathbf{p} \cdot \mathbf{p}$, $x^2 = \mathbf{x} \cdot \mathbf{x}$ is the 3-D dot product.

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[1] The Hilbert space usually considered in quantum mechanics is that of L^2 functions defined on some interval or higher-dimensional (open) subset U of euclidean space. L^2 means a Hilbert space complete under the L^2 norm, which is given by $\|f\|_2 = \int_U |f(x)|^2\, \mathrm{d}x$, where $\mathrm{d}x$ is the measure on U. The L^2 inner product is the familiar one: $f \cdot g = \int_U f^* g\, \mathrm{d}x$.

Exercise 4. In this problem, you will construct the 2×2 matrix corresponding to the component of spin angular momentum along an arbitrary direction $\hat{r}$.

a) A rotation can be specified by the Euler angles (α, β, γ), or by (θ, φ). The Euler angles represent first a rotation about $\hat{z}$ by an angle α, then a rotation *about the new* y-axis by an angle β, and then a rotation about the *new* z-axis again. Convince yourself that this works.

b) Now, suppose given a rotation specified by the Euler angles (α, β, γ). This is given in quantum mechanics by the matrix

$$e^{-i\gamma S_{z'}/\hbar} e^{-i\beta S_u/\hbar} e^{-i\alpha S_z/\hbar},$$

where the u-axis is the new y-axis after rotating about z, and the z'-axis is the new z-axis after rotating about $\hat{z}$ and $\hat{u}$. Show that this is the same matrix as

$$e^{-i\alpha S_z/\hbar} e^{-i\beta S_y/\hbar} e^{-i\gamma S_z/\hbar}.$$

Hint:[2]

c) Use part (b) with $S_i = \frac{\hbar}{2}\sigma_i$ to calculate the rotation matrix corresponding to placing the $\hat{z}$ axis along $\hat{r}$, where $\hat{r}$ is specified by the two angles (θ, φ). This is exactly the spin matrix S_r.

d) Calculate the eigenvalues and eigenspinors of S_r.

Exercise 5. A particle of mass m is placed in a finite spherical well

$$V(r) = \begin{cases} -V_0, & r \leq a \\ 0, & r \geq a \end{cases}.$$

Find the equation that quantizes the energy (you don't need to solve it), by solving the radial Schrödinger equation with $\ell = 0$. Explain how you could solve this equation and obtain the energies. Show that there is no bound state if $V_0 a^2 < \pi^2\hbar^2/8m$. Hint: [3]

Exercise 6. Show that Ehrenfest's theorem is valid in 3-dimensions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle = \frac{i}{\hbar}\langle [H, Q]\rangle + \left\langle \frac{\partial Q}{\partial t} \right\rangle.$$

Then, show that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}\rangle &= \frac{1}{m}\langle \mathbf{p}\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{p}\rangle &= \langle -\vec{\nabla} V\rangle. \end{aligned}$$

[2] Denoting a rotation about the axis r by an angle ζ as $R_r(\zeta)$, we have that $S_u = R_z(\alpha) S_y R_z(-\alpha) = e^{-i\alpha S_z/\hbar} S_y e^{i\alpha S_z/\hbar}$. Now, try to write a similar expression for $R_{z'}(\gamma) = e^{-i\gamma S_{z'}/\hbar}$.

[3] Recall that the radial Schrödinger equation is identical to the time-independent, 1-dimensional Schrödinger equation with the wavefunction replaced by $u(r) = rR(r)$ (where $\psi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi)$) and potential $V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$.